Traffic flow models by Markovian jump process

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joint work with

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Traffic flow models by Markovian jump process

→ Extracted from interacting particles systems widely studied in Probability and Theorical Physics (stationary state form and its stability, simulation methods)

→ A process describes the evolution in continuous time of particles jumping on a set of sites according to interaction rules

→ The systems can be investigated analytically with probabilistic tools and are easy to be simulated

→ The Markovian models represent an stochastic alternative to traffic flow modelling by differential system and an extension in continuous time of cellular automata approach

Presentation main lines

1. Principal caracteristics of a Markovian jump process
2. A microscopic traffic flow model defined on a discrete space by a *zero-range* process
3. A mesoscopic traffic flow model defined on a discrete space by a *misanthrope* process
4. Conclusion and working prospects

Main caracteristics of a Markovian jump process

$(X_t, t \in \mathbb{R}^+)$ defined on E is a homogeneous Markovian jump process if, for all n, all $0 \leq t_0 < t_1, \ldots < t_{n+1}$ and all $\eta_0, \ldots, \eta_{n+1} \in \mathsf{E}$ such as $\mathbb{P}(X_{t_0} = \eta_0, \ldots, X_{t_n} = \eta_n) \neq 0$, we have :

$$\begin{aligned} \mathbb{P}(X_{t_{n+1}} = \eta_{n+1} / X_{t_0} = \eta_0, \ldots, X_{t_n} = \eta_n) &= \mathbb{P}(X_{t_{n+1}} = \eta_{n+1} / X_{t_n} = \eta_n) \\ &= \mathcal{P}_{t_{n+1}-t_n}(\eta_n, \eta_{n+1}) \end{aligned}$$

$\rightarrow$ $\mathcal{P}_t$ is called transition matrix of the Markovian process (X_t) when E is finite or even only countable. The matrix are Markovian *i.e.* for all $\eta, \xi \in \mathsf{E}$:

$$\mathcal{P}_t(\eta, \xi) \geq 0, \qquad \sum_{k \in \mathsf{E}} \mathcal{P}_t(\eta, k) = 1$$

Main caracteristics of a Markovian jump process

$(X_t, t \in \mathbb{R}^+)$ is caracterised by its generator matrix L defined by :

$$\forall \eta \in \mathsf{E}, \quad \forall \xi \neq \eta \quad \mathsf{L}(\eta, \xi) = \left.\frac{\mathrm{d}\mathcal{P}_t}{\mathrm{d}t}\right|_{t=0} (\eta, \xi), \quad \mathsf{L}(\eta, \eta) = -\sum_{\xi} \mathsf{L}(\eta, \xi)$$

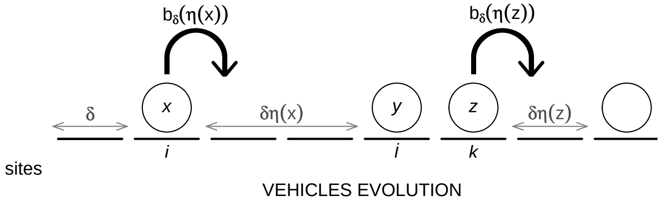
$\longrightarrow$ When E is *irreducible* (*i.e.* $\forall \eta, \xi, t, \ \mathcal{P}_t(\eta, \xi) > 0$) *reccurente* (*i.e.* for all $\eta \in \mathsf{E}$, leaving from η, the process will almost surely coming back in η), an invariante measure of the process $(X_t, t \in \mathbb{R}^+)$, denoted π, is solution of the equation :

$$\pi \mathsf{L} = 0 \qquad i.e. \qquad \forall \eta \in \mathsf{E} \quad \sum_{\xi \neq \eta} \pi(\xi) \mathsf{L}(\xi, \eta) = \sum_{\xi \neq \eta} \pi(\eta) \mathsf{L}(\eta, \xi)$$

$\longrightarrow$ A reversible measure μ, for which for all $\eta, \xi \in \mathsf{E}$ $\mu(\eta)\mathsf{L}(\eta, \xi) = \mu(\xi)\mathsf{L}(\xi, \eta)$, is invariante

Table of contents

A microscopic traffic flow model by a *zero-range* process



→ Vehicles evolves on a lane divided into cell of length δ. We considere vehicles distance gap $(\eta_t(v), t \in \mathbb{R}^+, v \in \mathbb{Z})$

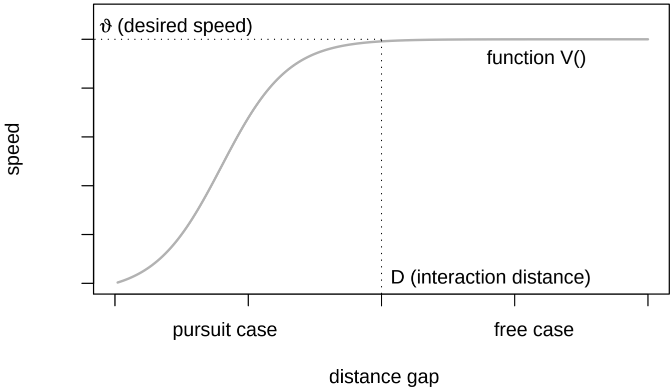
→ The rate of jump b_δ of a vehicle x is a function of its distance gap :

$$b_\delta(\eta_t(x)) = \frac{1}{\delta}\mathscr{V}(\eta_t(x) \times \delta)$$

$\mathscr{V}$ is a function of «targeted speed» depending on the distance gap

Targeted speed function parameter

Ones assumes the targeted speed function constant, equal to a desired speed denoted ϑ beyond an interaction distance $\mathcal{D}$



Process generator

$(\eta_t, t \in \mathbb{R}^+)$ is a zero-range process defined on $\mathsf{E} = \mathbb{N}^{\mathbb{Z}}$;
$\mathbb{Z}$ is the vehicle set, $\mathbb{N}$ is the vehicles distance gap discretised in unit δ

$\rightarrow$ The process is caracterised by the generator $\mathbf{L}$, defined for any function f by :

$$\mathbf{L}f(\eta) = \sum_x b_\delta(\eta(x))[f(\eta^x) - f(\eta)]\mathbb{1}_{\{\eta(x)>0\}}$$

$$\text{with} \begin{cases} \eta^x(z) & = & \eta(z) & \text{if } z \neq x \text{ and } z \neq x-1 \\ \eta^x(x) & = & \eta(x) - 1 \\ \eta^x(x-1) & = & \eta(x-1) + 1 \end{cases}$$

The jump rate only depends of the state of the departure site

Invariante distribution of the process on a infinite lane

$(\eta_t, t \in \mathbb{R}^+)$ is a zero-range process defined on $\mathsf{E} = \mathbb{N}^{\mathbb{Z}}$ whose asymptotic distribution is known on a finite and infinite space (SPITZER 70, ANDJEL 82)

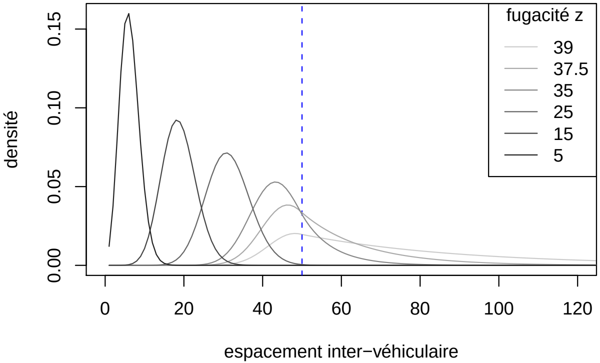
$\rightarrow$ For the case of an infinite lane, the $(\eta(x), x \in \mathbb{Z})$ are independent, identically distributed according to :

$$\tilde{\pi}_z(m) = \begin{cases} z^m \Psi_\delta(m)/C_\delta & \text{if } 0 \leq m < K \\ z^m \Psi_\delta(K-1)(\delta/\vartheta)^{m-K}/C_\delta & \text{if } m \geq K \end{cases}$$

with $K = \mathcal{D}/\delta$, $\Psi_\delta(m) = \prod_{n=1}^m (b_\delta(n))^{-1}$ with $\Psi_\delta(0) = 1$, $C_\delta = \sum_{m=0}^{K-1} z^m \Psi_\delta(m) + \frac{z^K \Psi_\delta(K)}{1-\delta z/\vartheta}$ and $z = \mathbb{E} b_\delta$ a parameter usually called *fugacity*

Distance gap distribution are uni-modal

→ That traduct the absence of kinematic (*stop-and-go*) waves



Calculus of performance indicators in stationary state

By construction of the model the mean speed is equal to $\mathcal{V} = \delta \times z$
$\longrightarrow$ Performance indicators depend of vehicles mean speed

The mean distance gap is deducted from the mean number of free cells in front :

$$\mathbb{E}\mathcal{D}_\delta(\mathcal{V}) = \frac{\delta}{C_\delta}\left(\sum_{m=0}^{K-1} m \prod_{n=1}^{m} \frac{\mathcal{V}}{\mathcal{V}(n\delta)} + \frac{K-1+\frac{1}{1-\mathcal{V}/\vartheta}}{1-\mathcal{V}/\vartheta} \prod_{n=1}^{K} \frac{\mathcal{V}}{\mathcal{V}(n\delta)}\right)$$

The flow density and flow volume are given by :

$$\begin{aligned} \varrho_\delta(\mathcal{V}) &= 1/(\mathbb{E}\mathcal{D}_\delta(\mathcal{V}) + \ell) \\ \mathcal{Q}_\delta(\mathcal{V}) &= \mathcal{V}/(\mathbb{E}\mathcal{D}_\delta(\mathcal{V}) + \ell) \end{aligned}$$

Calculus of performance indicators for the case $\delta \to 0$

The limit calculus $\delta \to 0$ allows to :

- Simplify the formulas;
- Evaluate the impact of the spatial discretisation

One shows that on $[0, \vartheta[$, $\mathbb{E}\mathcal{D}_\delta(\mathcal{V}) \to \mathscr{V}^{-1}(\mathcal{V})$ when $\delta \to 0$

By inversing the variables $\mathcal{V}$ and ϱ, one shows that, when $\delta \to 0$:

$$\mathcal{V}(\varrho) = \mathscr{V}(1/\varrho - \ell)$$
$$\mathcal{Q}(\varrho) = \varrho \times \mathscr{V}(1/\varrho - \ell).$$

$\to$ $\varrho^c = 1/(\mathcal{D} + \ell)$ is the critical density threshold beyond which vehicles mean speed become less that the desired speed

Study of the distance gap variance in stationary state

$$\mathbb{V}\mathcal{D}_\delta(z) = \delta^2 \left(\sum_m (\tilde{\pi}_z(m))^2 - \left(\sum_m \tilde{\pi}_z(m) \right)^2 \right)$$

Distance gap variance gives use an indicator of vehicles repartition

One show that :

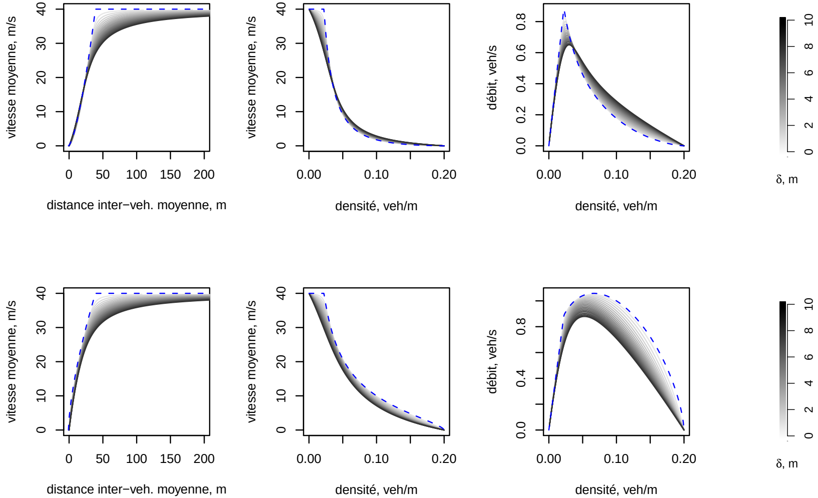
$$\forall\, \mathcal{V} \in [0, \vartheta[, \qquad \lim_{\delta \to 0} \mathbb{V}\mathcal{D}_\delta(\mathcal{V}) = 0$$

$\rightarrow$ This result and the previous expected value are obtained by showing that

$$\lim_{K \to \infty} \frac{\sum_{i=1}^{K-1} h\left(\frac{i}{K}\right) \prod_{j=1}^{i} \frac{g(d)}{g(j/K)}}{\sum_{i=1}^{K-1} \prod_{j=1}^{i} \frac{g(d)}{g(j/K)}} = h(d)$$

for all $d \in [0, 1]$ and under the asumptions $g \in C^1$ from $[0, 1]$ to $[0, 1]$ such as $0 < \alpha \leq g'$ and h with finite growth

Mean performances in the stationary state



Introduction of a reaction time parameter $\mathcal{T}^r$

$\longrightarrow$ At the instant t, the jump rate of the vehicle x becomes $\frac{1}{\delta}\mathscr{V}(\delta\eta_{t-\mathcal{T}^r}(x))$ where the delayed distance gap $\delta\eta_{t-\mathcal{T}^r}(x)$ is approximated by :

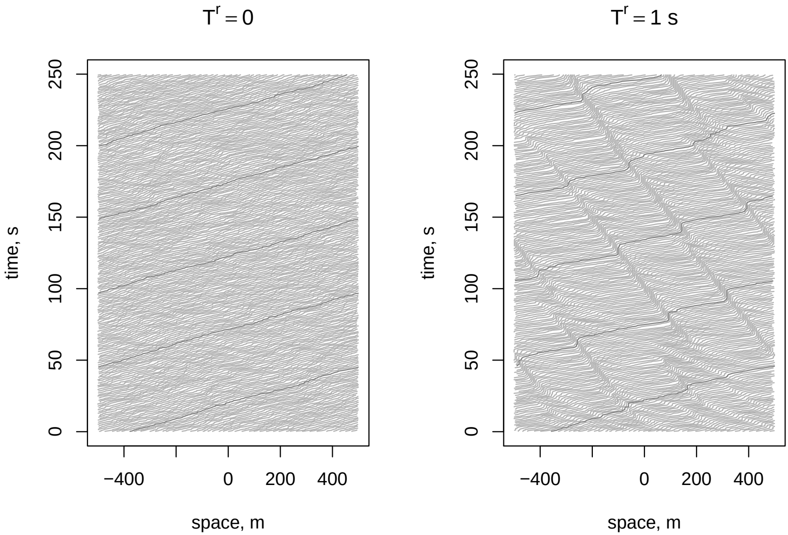
$$\delta \times \eta_{t-\mathcal{T}^r}(x) \approx \delta\eta_t(x) - \mathcal{T}^r(\mathscr{V}(\delta\eta_t(x+1)) - \mathscr{V}(\delta\eta_t(x)))$$

$\longrightarrow$ Process caracterised by the generator :

$$\mathbf{L}f(\eta) = \sum_x b_\delta(\eta(x), \eta(x+1))[f(\eta^x) - f(\eta)]\mathbb{1}_{\{\eta(x)>0\}}$$

$$\text{with} \left\{ \begin{array}{lcll} \eta^x(z) & = & \eta(z) & \text{if } z \neq x \text{ and } z \neq x-1 \\ \eta^x(x) & = & \eta(x) - 1 & \\ \eta^x(x-1) & = & \eta(x-1) + 1 & \end{array} \right.$$

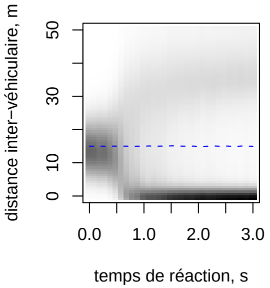
Example of vehicles trajectories on a ring



Monte-Carlo simulations on a ring

Phase transition from homogeneous (uni-modal) to heterogeneous (bi-modal) state with $\mathcal{T}^r$ (example $\vartheta = 40\, m/s$, $\mathscr{V}(d) = d$)

$\longrightarrow$ Condition observed : $\mathscr{V}' < 1/(2\mathcal{T}^r)$



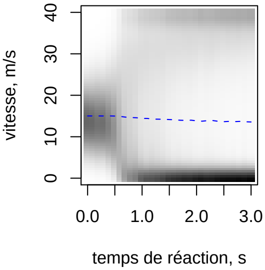
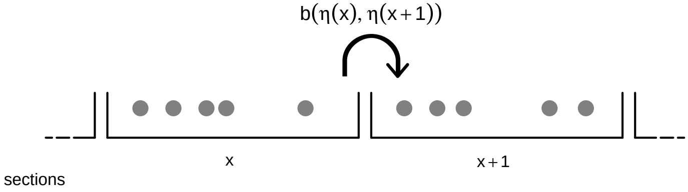


Table of contents

A mesoscopic traffic flow model by a *misanthrope* process



$\longrightarrow$ We considere the evolution of vehicles number $(\eta_t(x), t \in \mathbb{R}^+, x \in \mathbb{Z})$ by section of length D

$\longrightarrow$ The rate of jump of a vehicle from section x to section $x+1$ is :

$$b(\eta(x), \eta(x+1)) = \min \left\{ \Delta\left(\frac{\eta(x)}{D}\right), \Sigma\left(\frac{\eta(x+1)}{D}\right) \right\}$$

where Σ and Δ are respectively the function of supply and demand

Process generator

The jump rate function b is an increasing function of the vehicles number on the departure site and a decreasing function of the vehicles number on the arrival site

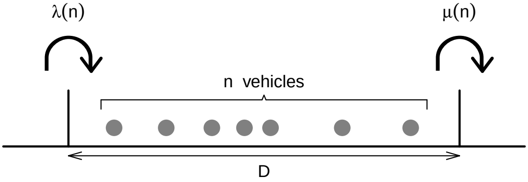
$\longrightarrow$ $(\eta_t, t \in \mathbb{R}^+)$ is a misanthrope process (Cocozza 85)

$\longrightarrow$ Caracterised by the generator :

$$\mathsf{L}f(\eta) = \sum_{x \in \mathbb{Z}} b(\eta(x), \eta(x+1))[f(\eta^x) - f(\eta)]\mathbb{1}_{\{\eta(x)>0\}}$$

with
$$\begin{cases} \eta^x(z) & = & \eta(z) & \text{if } z \neq x \text{ and } z \neq x+1 \\ \eta^x(x) & = & \eta(x) - 1 & \\ \eta^x(x+1) & = & \eta(x+1) + 1 & \end{cases}$$

Invariante state of one section with open boundaries



The enter rate λ and the exit rate μ are defined by :

$$\lambda(n) = \min\left\{\alpha, \Sigma\left(\frac{n}{D}\right)\right\} \quad \text{and} \quad \mu(n) = \min\left\{\Delta\left(\frac{n}{D}\right), \beta\right\}$$

where α is a demand upstream and β a supply downstream; the demand and supply functions are linear by piece :

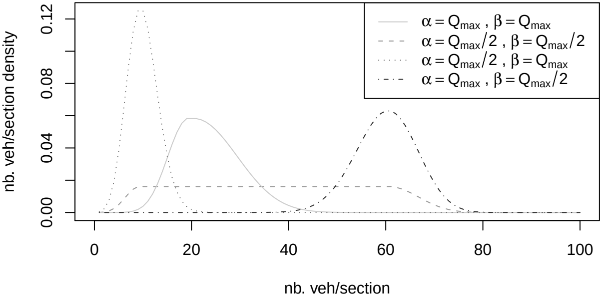
$$\Sigma\left(\frac{n}{D}\right) = \min\left\{\mathcal{Q}_{max}, a\left(\frac{n_{max}}{D} - \frac{n}{D}\right)\right\} \quad \text{and} \quad \Delta\left(\frac{n}{D}\right) = \min\left\{b\frac{n}{D}, \mathcal{Q}_{max}\right\}$$

Stationary distribution of vehicles number on the section

Since the process in reversible, the invariante measure π of the vehicles number on the section is solution of the equation :

$$\pi(n-1)\lambda(n-1) = \pi(n)\mu(n)$$

One finds $\pi(n) = \pi(0)\prod_{i=1}^{n} \frac{\lambda(i-1)}{\mu(i)}$, $\pi(0) = \left(1 + \sum_{n=1}^{n_{max}} \prod_{i=1}^{n} \lambda(i-1)/\mu(i)\right)^{-1}$

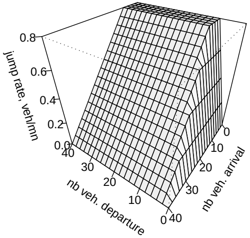


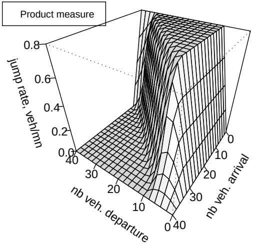
Product invariant distribution of a lane

To obtain a product invariant distribution, the jump rate function b has to satisfy the relations :

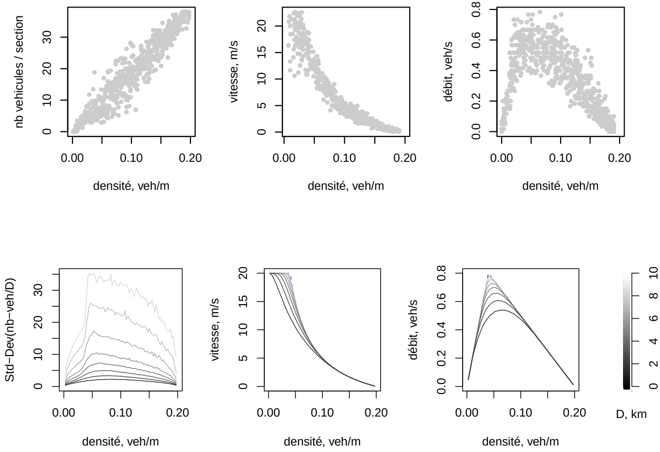
$$b(n,0) + \frac{b(1,n)}{b(n+1,0)} \frac{b(p,0)}{b(1,p-1)} b(n+1,p-1) = b(n,p) + b(p,0)$$

$\longrightarrow$ Numerical examples present unreasonable aspects

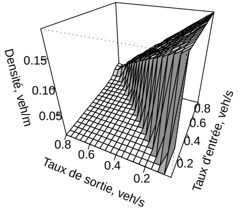


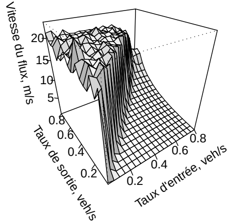


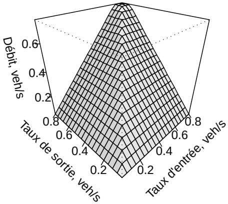
Performances in stationary state on a ring

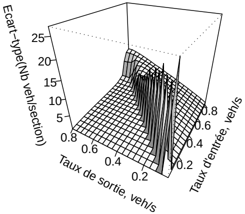


Stationary state performances of a lane with open boundaries

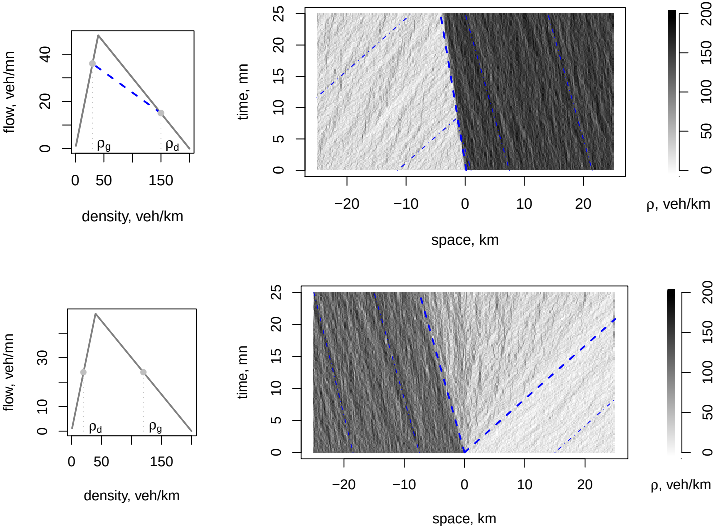






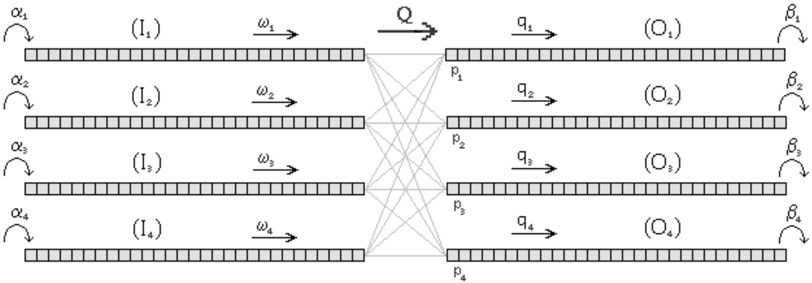


RIEMANN experiments



Towards the simulation of a network

Modelling of an intersection with multiple paths



Performances observed when $D \to \infty$:

Heterogeneous model

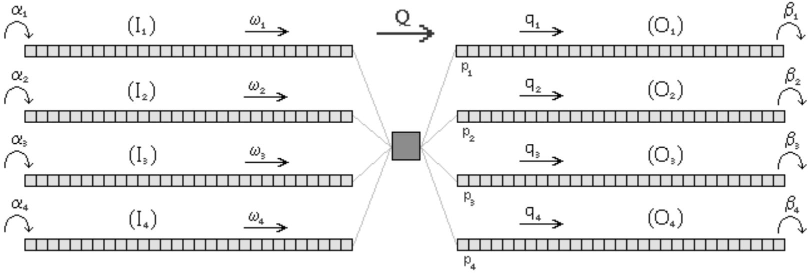
$$\begin{cases} q_d = \min\left\{p_d \times \sum_i \alpha_i, \beta_d\right\} \\ \mathcal{Q} = \sum_d q_d \end{cases}$$

FIFO model

$$\begin{cases} \mathcal{Q} = \min_d \left\{\sum_i \alpha_i, \frac{\beta_d}{p_d}\right\} \\ q_d = p_d \mathcal{Q} \end{cases}$$

Towards the simulation of a network

Modelling of an intersection with a bottleneck



Performances observed when $D \to \infty$:

Heterogeneous model
$$\begin{cases} q_d = \min\left\{p_d \times \min\left\{\sum_i \alpha_i, Q^0_{max}\right\}, \beta_d\right\} \\ Q = \sum_d q_d \end{cases}$$

FIFO model
$$\begin{cases} Q = \min_d\left\{\sum_i \alpha_i, Q^0_{max}, \frac{\beta_d}{p_d}\right\} \\ q_d = p_d Q \end{cases}$$

Table of contents

Conclusion and working prospects

Conclusion

→ Markovian jump models can reproduce several traffic flow caracteristics

→ They are polyvalent, easy to simulate and can be investigated analytically (at least in basic cases)

Working prospects

→ Stationary state calculus using product and n-clusters forms

→ Explicite conditions on microscopic models parameters of the transition phase uni-modal / bi-modal states

→ Study of a continuous space model using a *Totally Asymmetric Random Average Process*