Properties in stationary state of a microscopic traffic model mixing stochastic transport and car-following

Sylvain LASSARRE\textsuperscript{a}  Michel ROUSSIGNOL\textsuperscript{b}  Andreas SCHADSDHNEIDER\textsuperscript{c}  Antoine TORDEUX\textsuperscript{d}

\textsuperscript{a} Institut Français des Sciences et Technologies des Transports, de l’Aménagement et des Réseaux (IFSTTAR)  
\textsuperscript{b} Paris-Est University – Laboratoire Analyse et Mathématiques Appliquées  
\textsuperscript{c} Cologne University – Institut für Theoretische Physik  
\textsuperscript{d} Paris-Est University – Laboratoire Ville Mobilité Transport

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Objective and outline

Model uni-dimensional vehicle line using

- Parameters of car-following model by delay-differential equation
  → speed function of the gap, driver reaction time
- Stochastic transport process by markovian jump process
  → Continuous extension of Cellular Automata models
  → Interacting Particle Systems statistical physic theory

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Table of contents

- Real traffic experiment and model parameters
- Description of the basic stochastic car-following model
- Description of the model including a reaction time
- Conclusion
Behavior of a vehicle line for interactive density levels: kinematic (stop-and-go) waves

Real vehicle trajectories on a ring → Bi-modal vehicle performances

Corresponding Space-Time diagram

Modelling assumptions and parameters

- Coupled interaction between the vehicle and the predecessor → Speed $V$ as a function of the distance gap

- Delay in the regulation process of a driver → Striclty positive reaction time parameter $T'$
INTRODUCTION

BASIC STOCHASTIC CAR-FOLLOWING MODEL

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**Definition of the model 1**

**Deterministic**

- **NEWELL** first-order car-following model\(^a\) with no delay
  \[ \dot{x}_i(t) = V(x_{i+1}(t) - x_i(t)) \]
  \(\rightarrow\) Stable if \(V' > 0\)

- Explicit Euler discretisation scheme with time step \(\delta t > 0\) gives
  \[ x_i(t + \delta t) = x_i(t) + \delta t \times V(x_{i+1}(t) - x_i(t)) \]
  \(\rightarrow\) Vehicles jump synchronously of \(\delta t \times V\) at each time step \(\delta t\)


**Stochastic**

- Each vehicle jump independently according to a homogeneous poissonian process of parameter \(1/\delta t\)
  - Jump size of the vehicle \(i\) at \(t\) is \(\delta t \times V(x_{i+1}(t) - x_i(t))\)
Mathematical formulation

- \( N \geq 2 \) vehicles on a ring of length \( L \) with curvilinear abscissa \((x_i)_i\)
  with \( i \in [1, N] \) the index of the vehicles (vehicle \( i + 1 \) is the predecessor of the vehicle \( i \))

- \( \Delta_i = x_{i+1} - x_i \), for \( i < N \), and \( \Delta_N = x_1 - x_N + L \) are the vehicle spacings defined on \( \mathbb{R}_+ \)

- \( \eta = (\Delta_i)_i \) is a markovian jump process defined on \( E = \mathbb{R}_+^N \)

- Process characterised by the generator \( \mathcal{L} \) given for any function \( f : E \mapsto \mathbb{R} \) by

\[
\mathcal{L} f(\eta) = \sum_i \frac{1}{\delta t} [f(\eta^i) - f(\eta)] \mathbb{1}_{\{\delta t \leq \Delta_i / V(\Delta_i)\}}
\]

with \( \eta^i = (\Delta^i_j)_j \) and \( \Delta^i_j = \begin{cases} 
\Delta_j & \text{if } j \neq i, i - 1 \\
\Delta_i - \delta t \times V(\Delta_i) & \text{if } j = i \\
\Delta_{i-1} + \delta t \times V(\Delta_i) & \text{if } j = i - 1
\end{cases} \)
Link with the TARAP

• If \( V(d) = d/T \), \( \eta \) is a Totally Asymmetric Random Average Process\(^a\)

\[
\mathcal{L}f(\eta) = \sum_i \frac{1}{\delta t} \int p(u) \, du \, [f(\eta^i(u)) - f(\eta)] \mathbf{1}_{\{\Delta_i > 0\}}
\]

with \( p \) a pdf on \([0, 1]\), and \( \Delta^i_j(u) = \begin{cases} 
\Delta_j \\
u\Delta_i \\
\Delta_{i-1} + (1-u)\Delta_i 
\end{cases} 
\)

if \( j \neq i, i-1 \)

if \( j = i \)

if \( j = i-1 \)

\( \rightarrow p \) is deterministic with the model 1: \( p(u) = \delta_{1-\delta t/T}(u) \) \((0 < \delta t \leq T)\)

\(^a\)M. ROUSSIGNOL, Ann. Inst. Henri Poincaré B 16(2), 101–108

• In the infinite case \( L = \infty \), if initial system distribution \( \alpha \in E \) is homogeneous in space with \( E\alpha_i = D \) and \( \sum_i |E\alpha_1 \alpha_i - D^2| < \infty \), then, denoting \( r = \int (1-u)p(u) \, du \) and \( s = \int u(1-u)p(u) \, du \)

\[
E\Delta_i = D \quad \forall i \forall t \quad \text{and} \quad \lim_{t \to \infty} \text{var} \Delta_i = D^2(r/s - 1) \quad \forall i \\
= \delta t D^2/(T - \delta t)
\]
Definition of the model 2

- Vehicle jump size is random
- \( p \) has a beta distribution on \([0, 1]\) with parameters \( m, n > 0 \)

\[
p(u) = \frac{1}{\beta(m, n)} u^{m-1}(1-u)^{n-1} \mathbb{1}_{[0,1]}(u)
\]

with \( \beta(m, n) = \int_0^1 u^{m-1}(1-u)^{n-1} \, du \)

\( \rightarrow \) One denotes \( m = (1 - \delta t / T)(K - 1) \) and \( n = \delta t (K - 1) / T \) with \( K > 1 \) calibrating the variability of \( p \)

- For the infinite system, the spacing variability in stationary state is

\[
\lim_{t \to \infty} \nabla \Delta_i = D^2 \frac{T + \delta t (K - 1)}{(T - \delta t)(K - 1)} \quad \forall i
\]

\( \rightarrow \)

\[
\delta t \to 0 \quad D^2 / (K - 1)
\]
Invariant distribution calculus

- The invariant distribution $\pi : E \mapsto [0, 1]$ satisfies
  \[
  \int_E \pi(d\eta) \mathcal{L}f(\eta) = 0 \tag{1}
  \]

- If $\pi$ admits a product density form: $\pi(d\eta) = \prod_i \tilde{\pi}(\Delta_i) \prod_j d\Delta_j$, with marginal $\tilde{\pi} : \mathbb{R}^+ \mapsto [0, 1]$, (1) holds if $\forall i$
  \[
  \int_0^{\Delta_i-1} p \left( \frac{\Delta_i}{x + \Delta_i} \right) \frac{dx}{x + \Delta_i} \tilde{\pi}(\Delta_i - x) \tilde{\pi}(\Delta_i + x) = \tilde{\pi}(\Delta_i-1)\tilde{\pi}(\Delta_i) \tag{2}
  \]

- If $\tilde{\pi}$ is gamma distributed: $\tilde{\pi}(x) = x^{\gamma-1} \exp(-x/\theta) \frac{\Gamma(\gamma)}{\Gamma(\gamma+\theta)} \mathbb{1}_{[0,\infty)}(x)$ with $\Gamma(\gamma) = \int_0^\infty u^{\gamma-1} e^{-u} du$, $\gamma = K - 1$ and $\theta = (K - 1)/D$, (2) leads to
  \[
  \left( \frac{\Delta_i-1}{\Delta_i} \right)^n \frac{(\Gamma(\gamma))^2}{\Gamma(\gamma+n)\Gamma(\gamma-n)} = 1
  \]
  that holds if $n \to 0$ that is $\delta t \to 0$ or $K \to 1$
Empirical invariant marginal spacing distribution

→ Distributions tend towards the gamma form when $\delta t \to 0$
Examples of trajectories on a ring

$K = 2$

$K = 50$
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Definition of the model 3

Deterministic

- **NEWELL** car-following model with delay $T' \geq 0$, denoting
  \[ \Delta_i = x_{i+1} - x_i \]
  \[ \dot{x}_i(t) = V(\Delta_i(t - T')) \]
  \[ \rightarrow \text{Stable if } 0 < V' < 1/(2T') \]

- Explicit Euler discretisation scheme with time step $\delta t > 0$ and linear approximation for $T'$ gives
  \[ x_i(t + \delta t) = x_i(t) + \delta t \times V[\Delta_i(t) - T'(\dot{x}_{i+1}(t) - \dot{x}_i(t))] \]

Stochastic

- Each vehicle jump independently according to a homogeneous Poissonian process of parameter $1/\delta t$

- Jump size of the vehicle $i$ at $t$ is deterministic:
  \[ s_i(t, T', \delta t) = \delta t \times V[\Delta_i(t) - T'(V(\Delta_{i+1}(t)) - V(\Delta_i(t)))] \]
Mathematical formulation

- $\eta = (\Delta_i)_i$ is a markovian jump process defined on $E = \mathbb{R}_+^N$

- Process characterised by the generator $\mathcal{L}$ given for any function $f : E \mapsto \mathbb{R}$ by

$$\mathcal{L} f(\eta) = \sum_i \frac{1}{\delta t} [f(\eta^i) - f(\eta)] \mathbb{1}_{\{\Delta_i \leq s_i\}}$$

with $\eta^i = (\Delta^i_j)_j$ and $\Delta^i_j = \begin{cases} 
\Delta_j & \text{if } j \neq i, i - 1 \\
\Delta_i - s_i & \text{if } j = i \\
\Delta_{i-1} + s_i & \text{if } j = i - 1 
\end{cases}$

- This process is not a known markovian jump one: mass is transfer from a site to the following one while jump rate depends of considered and preceding sites

- Stationary distribution hard to calculate analytically, it may be investigated by simulation
Examples of trajectories on a ring

$T_r = 0$

$T_r = 1$
For interactive density level and sufficiently high reaction time, the distributions tend towards bi-modal ones when $\delta t \to 0$.

Condition for stop-and-go wave emergence is the same as the stability condition of the deterministic car-following models:

$$T' > 1/(2V') \quad (= T/2)$$

Maximum vehicle speed specifies critical density level (free or congested)
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Conclusion

Theoretical (model 1 and 2 with no reaction time)

- Asymptotic invariant distributions are obtained for a totally asymmetric random average process

Practical (model 3 with reaction time)

- Stop-and-go wave emergence conditions are observed with different ways of modelling (deterministic by car-following model and stochastic by markovian jump process)
  ⇝ Fundamental mecanism between « reaction time » and « targeted speed function form »
Thank you for your attention