Properties in stationary state of a microscopic traffic model mixing stochastic transport and car-following

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Objective and outline

Model uni-dimensional vehicle line using

- Parameters of car-following model by delay-differential equation
  → speed function of the gap, driver reaction time
- Stochastic transport process by markovian jump process
  → Continuous extension of Cellular Automata models
  → Interacting Particle Systems statistical physic theory[a]

[a] T.M. LIGGETT, *Interacting Particle Systems*, Springer 1985

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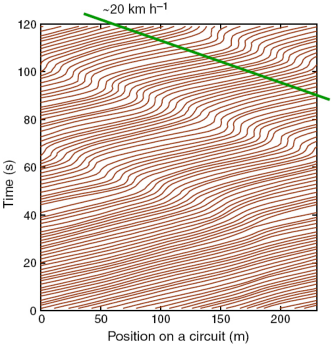
Behavior of a vehicle line for interactive density levels : kinematic (stop-and-go) waves

(movie)

Real vehicle trajectories on a ring

Corresponding Space-Time diagram[1]

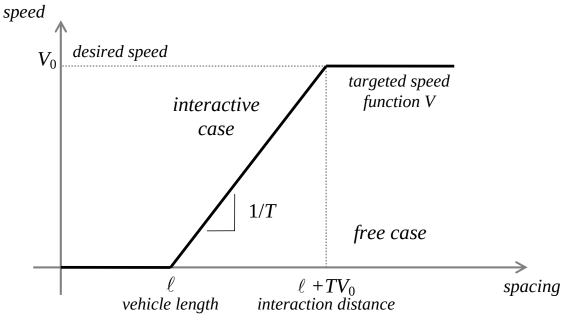


→ Bi-modal vehicle performances

[1] SUGIYAMA *et al.*, *New J. Phys.* **10** (2008) 033001

Modelling assumptions and parameters

- Coupled interaction between the vehicle and the predecessor
  $\longrightarrow$ Speed V as a function of the distance gap



- Delay in the regulation process of a driver
  $\longrightarrow$ StricIty positive reaction time parameter T^r

Definition of the model 1

Simulation

Deterministic

- NEWELL first-order car-following model[a] with no delay

$$\dot{x}_i(t) = V(x_{i+1}(t) - x_i(t))$$

$\longrightarrow$ Stable if $V' > 0$

- Explicite Euler discretisation scheme with time step $\delta t > 0$ gives

$$x_i(t + \delta t) = x_i(t) + \delta t \times V(x_{i+1}(t) - x_i(t))$$

$\longrightarrow$ Vehicles jump synchronously of $\delta t \times V$ at each time step δt

[a] G.F. NEWELL, *Op. Res.* **36(3)** (1961) 195–205

Stochastic

- Each vehicle jump independently according to a homogeneous poissonian process of parameter $1/\delta t$
- Jump size of the vehicle i at t is $\delta t \times V(x_{i+1}(t) - x_i(t))$

Mathematical formulation

- $N \geq 2$ vehicles on a ring of length L with curvilinear abscissa $(x_i)_i$ with $i \in [\![1, N]\!]$ the index of the vehicles (vehicle $i+1$ is the predecessor of the vehicle i)
- $\Delta_i = x_{i+1} - x_i$, for $i < N$, and $\Delta_N = x_1 - x_N + \mathrm{L}$ are the vehicle spacings defined on $\mathbb{R}_+$
- $\eta = (\Delta_i)_i$ is a markovian jump process defined on $E = \mathbb{R}_+^N$
- Process characterised by the generator $\mathscr{L}$ given for any function $f : E \mapsto \mathbb{R}$ by

$$\mathscr{L}f(\eta) = \sum_i \frac{1}{\delta t}[f(\eta^i) - f(\eta)]\mathbb{1}_{\{\delta t \leq \Delta_i / V(\Delta_i)\}}$$

with $\eta^i = (\Delta_j^i)_j$ and $\Delta_j^i = \begin{cases} \Delta_j & \text{if } j \neq i, i-1 \\ \Delta_i - \delta t \times V(\Delta_i) & \text{if } j = i \\ \Delta_{i-1} + \delta t \times V(\Delta_i) & \text{if } j = i-1 \end{cases}$

Link with the TARAP

- If $V(d) = d/T$, η is a Totally Asymmetric Random Average Process[a]

$$\mathscr{L}f(\eta) = \sum_i \frac{1}{\delta t} \int p(u)\, \mathrm{d}u\, [f(\eta^i(u)) - f(\eta)]\mathbb{1}_{\{\Delta_i > 0\}}$$

with p a pdf on $[0, 1]$, and $\Delta_j^i(u) = \begin{cases} \Delta_j & \text{if } j \neq i, i-1 \\ u\Delta_i & \text{if } j = i \\ \Delta_{i-1} + (1-u)\Delta_i & \text{if } j = i-1 \end{cases}$

$\rightarrow$ p is deterministic with the model 1: $p(u) = \delta_{1-\delta t/T}(u)$ $(0 < \delta t \leq T)$

[a] M. ROUSSIGNOL, *Ann. Inst. Henri Poincaré B* **16(2)**, 101–108

- In the infinite case $L = \infty$, if initial system distribution $\alpha \in E$ is homogeneous in space with $\mathbb{E}\alpha_i = D$ and $\sum_i |\mathbb{E}\alpha_1\alpha_i - D^2| < \infty$, then, denoting $r = \int (1-u)p(u)\,\mathrm{d}u$ and $s = \int u(1-u)p(u)\,\mathrm{d}u$

$$\mathbb{E}\Delta_i = D \quad \forall i \forall t \qquad \text{and} \qquad \lim_{t\to\infty} \operatorname{var} \Delta_i = D^2(r/s - 1) \quad \forall i$$
$$= \delta t\, D^2/(T - \delta t)$$

Definition of the model 2

Simulation

- Vehicle jump size is random
- p has a beta distribution on $[0,1]$ with parameters $m, n > 0$

$$p(u) = \frac{1}{\beta(m,n)} u^{m-1}(1-u)^{n-1} \mathbb{1}_{[0,1]}(u)$$

with $\beta(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\,\mathrm{d}u$

$\rightarrow$ One denotes $m = (1 - \delta t/T)\,(K-1)$ and $n = \delta t\,(K-1)\,/\,T$ with $K > 1$ calibrating the variability of p

- For the infinite system, the spacing variability in stationary state is

$$\lim_{t\to\infty} \mathbb{V}\Delta_i \quad = \quad D^2 \frac{T+\delta t(K-1)}{(T-\delta t)(K-1)} \quad \forall i$$
$$\underset{\delta t \to 0}{\longrightarrow} \quad D^2/(K-1)$$

Invariant distribution calculus

- The invariant distribution $\pi : E \mapsto [0,1]$ satisfies

$$\int_E \pi(\mathrm{d}\eta)\mathscr{L}f(\eta) = 0 \tag{1}$$

- If π admits a product density form: $\pi(\mathrm{d}\eta) = \prod_i \tilde{\pi}(\Delta_i)\prod_j \mathrm{d}\Delta_j$, with marginal $\tilde{\pi} : \mathbb{R}^+ \mapsto [0,1]$, (1) holds if $\forall i$

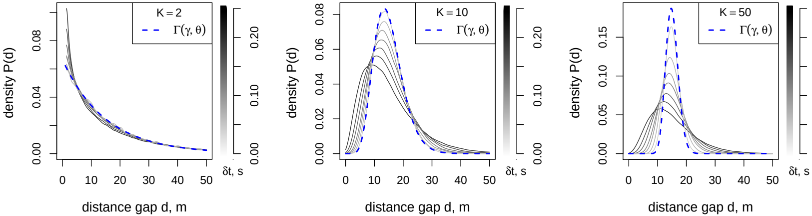
$$\int_0^{\Delta_{i-1}} p\left(\frac{\Delta_i}{x+\Delta_i}\right)\frac{\mathrm{d}x}{x+\Delta_i}\tilde{\pi}\left(\Delta_{i-1}-x\right)\tilde{\pi}\left(\Delta_i+x\right) = \tilde{\pi}(\Delta_{i-1})\tilde{\pi}(\Delta_i) \tag{2}$$

- If $\tilde{\pi}$ is gamma distributed: $\tilde{\pi}(x) = x^{\gamma-1}\frac{\exp(-x/\theta)}{\Gamma(\gamma)\theta^{\gamma}}\mathbb{1}_{[0,\infty)}(x)$ with $\Gamma(\gamma) = \int_0^\infty u^{\gamma-1}e^{-u}\mathrm{d}u$, $\gamma = K-1$ and $\theta = (K-1)/D$, (2) leads to

$$\left(\frac{\Delta_{i-1}}{\Delta_i}\right)^n \frac{(\Gamma(\gamma))^2}{\Gamma(\gamma+n)\Gamma(\gamma-n)} = 1$$

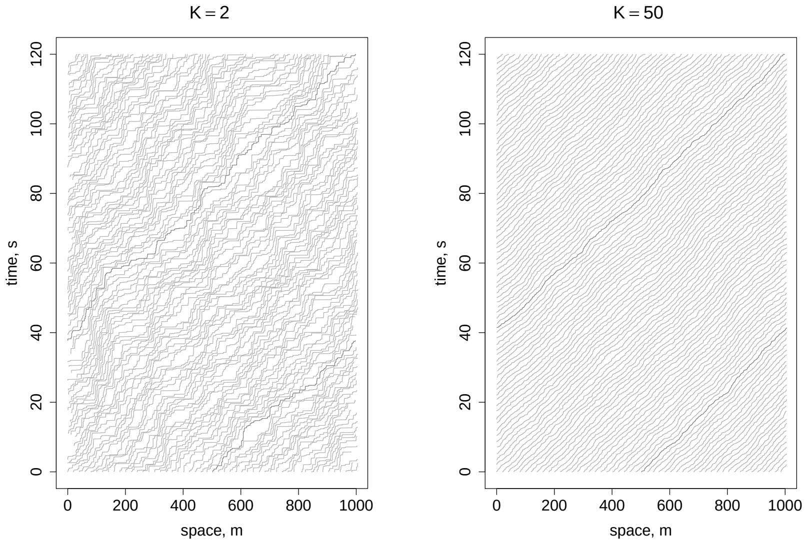
that holds if $n \to 0$ that is $\delta t \to 0$ or $K \to 1$

Empirical invariant marginal spacing distribution



→ Distributions tend towards the gamma form when $\delta t \to 0$

Examples of trajectories on a ring



Definition of the model 3

Simulation

Deterministic

- Newell car-following model with delay $\mathrm{T^r} \geq 0$, denoting $\Delta_i = x_{i+1} - x_i$

$$\dot{x}_i(t) = V(\Delta_i(t - \mathrm{T^r}))$$

$\rightarrow$ Stable if $0 < V' < 1/(2\,\mathrm{T^r})$

- Explicite Euler discretisation scheme with time step $\delta t > 0$ and linear approximation for $\mathrm{T^r}$ gives

$$x_i(t + \delta t) = x_i(t) + \delta t \times V\big[\Delta_i(t) - T^r(\dot{x}_{i+1}(t) - \dot{x}_i(t))\big]$$

Stochastic

- Each vehicle jump independently according to a homogeneous poissonian process of parameter $1/\delta t$
- Jump size of the vehicle i at t is deterministic:

$$s_i(t, \mathrm{T^r}, \delta t) = \delta t \times V\big[\Delta_i(t) - T^r(V(\Delta_{i+1}(t)) - V(\Delta_i(t)))\big]$$

Mathematical formulation

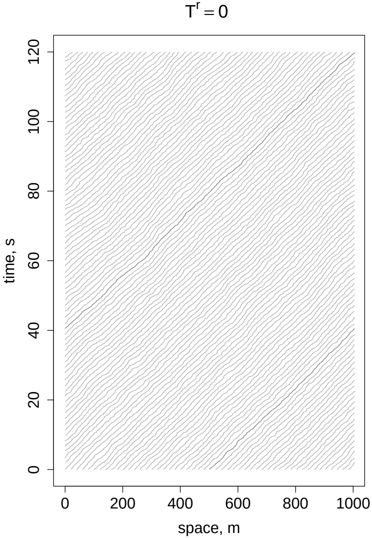
- $\eta = (\Delta_i)_i$ is a markovian jump process defined on $E = \mathbb{R}_+^N$
- Process characterised by the generator $\mathscr{L}$ given for any function $f : E \mapsto \mathbb{R}$ by

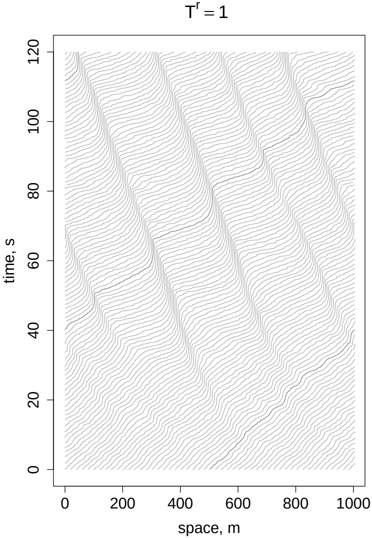
$$\mathscr{L} f(\eta) = \sum_i \frac{1}{\delta t}[f(\eta^i) - f(\eta)]\mathbb{1}_{\{\Delta_i \leq s_i\}}$$

with $\eta^i = (\Delta^i_j)_j$ and $\Delta^i_j = \begin{cases} \Delta_j & \text{if } j \neq i, i-1 \\ \Delta_i - s_i & \text{if } j = i \\ \Delta_{i-1} + s_i & \text{if } j = i-1 \end{cases}$

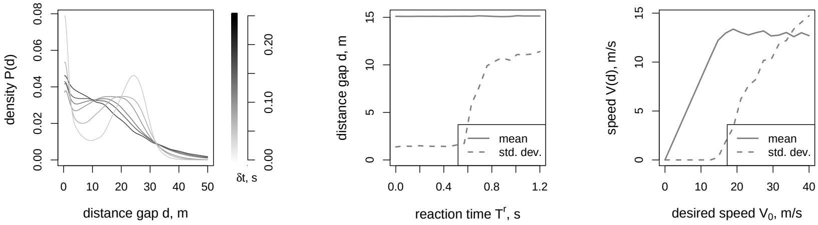
- This process is not a known markovian jump one: mass is transfer from a site to the following one while jump rate depends of considered and preceding sites
- Stationary distribution hard to calculate analytically, it may be investigated by simulation

Examples of trajectories on a ring





Empirical invariant marginal spacing distribution



→ For interactive density level and sufficiently high reaction time, the distributions tend towards bi-modal ones when $\delta t \to 0$

→ Condition for stop-and-go wave emergence is the same as the stability condition of the deterministic car-following models:

$$\mathrm{T}^{\mathrm{r}} > 1/(2V') \quad (= T/2)$$

→ Maximum vehicle speed specifies critical density level (free or congested)

Conclusion

Theorical (model 1 and 2 with no reaction time)

- Asymptotic invariant distributions are obtained for a totally asymmetric random average process

Practical (model 3 with reaction time)

- Stop-and-go wave emergence conditions are observed with different ways of modelling (deterministic by car-following model and stochastic by markovian jump process)

  $\rightsquigarrow$ Fundamental mecanism between « reaction time » and « targeted speed function form »

Thank you for your attention